Юбилейна международна научна конференция „Синергетика и рефлексия в обучението по математика“, 16-18 октомври 2020 г., Пампорово, България
Anniversary International Scientific Conference “Synergetics and Reflection in Mathematics Education”, 16-18 October 2020, Pamporovo, Bulgaria

POLYGONS FOR THE SECONDARY SCHOOL

Boyko Bantchev[1,*]

[1,*] *Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., bl.8, 1113 Sofia, bantchev@math.bas.bg*

Abstract. The text aims to demonstrate that essential knowledge on general polygons can be presented in a manner accessible to secondary school students. The topics addressed include the Jordan theorem for polygons, orientation, convexity of vertices, and triangulation.

Key Words: *polygons, Jordan theorem, orientation, convex vertex, triangulation.*

Introduction

Polygons are strange inhabitants of our mathematical culture. They are a relatively simple kind of planar figures that nevertheless exhibit remarkable structural versatility and possess a number of interesting properties. Their practical value is immense: applications abound in mathematical and mechanical modeling as well as in computing. Polygons are a rich and important domain of algorithm design in computational geometry.

Yet, general polygons are nearly absent from mathematical education. Apart from triangles and convex quadrilaterals, polygons are not studied in secondary school. The same largely holds for college and university education, with only a small number of partial exceptions.

The unfortunate lack of attention for polygons in school may be due to the unavailability of a useful, simple, systematic, and mathematically rigorous approach to presenting the subject. Perhaps some of the core knowledge pertaining to polygons is deemed too abstract or in other ways difficult to deal with in school.

We believe that such difficulties, if any, are perceived rather than real. In defending this standpoint, the rest of this text is dedicated to presenting, in a manner accessible to secondary school students, several essential topics related to polygons.

The Jordan curve theorem for polygons

A plane curve is called simple if it does not intersect itself. A Jordan curve is one that is both simple and closed. The Jordan curve theorem states that any Jordan curve C divides the set of points in the plane, from which C itself is excluded, into two components, not connected with each other but each one connected and each having C as its boundary. One of the sets is bounded and considered *interior*, and the other is unbounded and considered *exterior* with respect to C.

The theorem is famous for stating a seemingly obvious fact that is notoriously difficult to prove. Indeed, all known self-contained proofs are each no less than several pages long. Even in the particular case when C is a polygonal curve, a proof of the theorem, though considerably simpler, is usually still fairly complicated. In fact, what the theorem asserts is not as trivial as it would seem at first, because a 'simple' curve, be it a polygonal one, can have a myriad of meanders and roundabouts.

We offer here a proof of the Jordan curve theorem for the polygonal case that is, hopefully, more intuitive than the known ones, and that uses induction. The use of induction on the number of vertices of a polygon is natural and a good opportunity to combine geometric and combinatorial reasoning in solving a problem. In the course of presenting the proof of the theorem we establish the validity of several accompanying propositions.

Let C be a simple closed polyline. A straight line is said to be supporting to C if it contains one or more points of C and there is a half-plane with respect to the line that contains no points of C. If a direction is associated with the line and e.g. the right half-plane with respect to that direction is the one that contains no points of C, we say that the line supports C from the right, or is a right supporting line.

Now, whatever direction we choose in the plane, there is a right supporting line to C with that direction. Indeed, let us take an arbitrary line l of the chosen direction. If its right half-plane does not contain points from C, a right supporting line is the one parallel to l and passing through the nearest vertex of C to l (including l itself if the nearest vertex of C is on l). If the right half-plane of l does contain points from C, a right supporting line is the one parallel to l and passing through the farthest vertex of C on the right of l.

We note that, by construction, the supporting line contains at least one vertex of C. Also, if the line contains more than one point of C, then among these points there are at least two vertices: assuming otherwise would mean that the line intersects C at a non-vertex point, and that contradicts with it being a supporting line. Furthermore, it is possible to choose, among the infinitely many directions, one such that the supporting line has only a single vertex on it. If it was not so, and if no vertex lies on infinitely many supporting lines, then there must be infinitely many vertices – a contradiction. And if some vertex lies on infinitely many supporting lines, always accompanied by another vertex, then again there must be infinitely many vertices.

The points in the right half-plane of a right supporting line form an unbound set and therefore should belong (together with other points) to C's exterior. Keeping this in mind, we are prepared to prove the theorem.

Let the vertices of C, in some order along C, be $V_1,V_2,\ldots,V_n$. If $n = 3$ – C is a triangle – the theorem obviously holds, establishing the base of the induction hypothesis.

Let n be greater than 3. We construct a right supporting line l passing through a single point of C which by necessity is a vertex. Without loss of generality we can assume that vertex to be V_1. If there are no points of C inside $\Delta V_nV_1V_2$ or on V_nV_2 (other than V_1, V_2 and V_n), we mark the line segment V_nV_2. If there are points of C inside $\Delta V_nV_1V_2$, then this triangle is sure to contain a vertex (otherwise C would be self-intersecting, which it is not). In general, there may be more than one such vertex, so let V_i be the one farthest from V_nV_2. And if there are no points of C inside $\Delta V_nV_1V_2$ but there are such points on V_nV_2 (between its end points), then again there is necessarily at least one vertex among them, so let V_i be such a vertex. We then mark the line segment V_1V_i. Note that in either case the marked segment is guaranteed to not contain points of C other than its end points.

Now we use the marked segment, V_nV_2 or V_1V_i, to form two polygons, either $\Delta V_nV_1V_2$ and $V_nV_2\ldots V_{n-1}$ or $V_1V_2\ldots V_i$ and $V_1V_i\ldots V_n$. In either case the marked segment is common to the two new polygons and each of them has only a part of the vertices $V_1,V_2, \ldots ,V_n$, thus less than n in total. By induction hypothesis, each such polygon divides the plane as stated by the theorem. In order to finally prove the theorem, we must demonstrate that from it holding for the two smaller polygons it follows that it also holds for C.

Let A^+ , A^- and A be the interior, exterior, and the boundary of one of the polygons, similarly B^+ , B^- and B for the other, and E be the marked segment. We need to show that $C^+ = A^+ \cup B^+ \cup E$ is the interior of C, $C^- = A^- \cap B^-$ is the exterior, and that they are separated by $C = (A \setminus E) \cup (B \setminus E)$.

Indeed, A^+ and B^+ are connected within themselves by induction hypothesis, and so is E because it does not contain points of C; each of A^+ and B^+ is obviously connected to E; and each of A^+ and B^+ is connected to the other through E. Thus C^+ is connected. However, it is not connected to any point in C^-. For example, a point of A^+ is, with respect to A, not connected to any point of C^- because those are points of A^-. So with respect to C they could only have been connected through E but that also means that points of B^+ are directly connected to C^- and therefore to B^- which the induction hypothesis forbids. Finally, C^- is connected because each of A^- and B^- is, with respect to A and B, and to not be connected with respect to C would mean that the former connectedness is but through $A^+ \cup A \setminus E$ or $B^+ \cup B \setminus E$ which is absurd.

Thus the Jordan curve theorem is proved for polygons.

Orientation and complexity

Once the theorem is established, a number of useful consequences follow.

For example, since we can distinguish between interior and exterior, attaching orientation to C is meaningful. By convention, positive orientation is associated with the interior being on the left and the exterior on the right of C. Thus the polygon is positively oriented if interior is on the left when visiting the vertices in the (cyclic) order $V_1V_2\ldots V_n$. In particular, it follows that a triangle ABC is positively oriented if AC is in the left half-plane with respect to the oriented line AB.

One way to know whether a general polygon is positively oriented is by using a supporting line. If a right supporting line passes through a vertex V_i, C is positively or negatively oriented according as the order induced by the line on C is $V_{i-1}V_iV_{i+1}$ or $V_{i+1}V_iV_{i-1}$. We can observe that the orientation of C in increasing order of indices is the same as that of $\Delta V_{i-1}V_iV_{i+1}$. Furthermore, all vertices V_i for which $\Delta V_{i-1}V_iV_{i+1}$ is oriented the same way as C are the convex vertices of C, and those for which $\Delta V_{i-1}V_iV_{i+1}$ has opposite orientation are non-convex. Conversely, we can tell the orientation of a polygon by knowing only three consecutive vertices V_{i-1}, V_i, V_{i+1} and whether V_i is convex or not.

These are useful ways to relate orientation and convexity but of course, the convexity of a vertex does not *depend* on orientation and can be *defined* e.g. as follows: a vertex V_i is convex if in a sufficiently small vicinity of V_i the smaller angle between the rays V_iV_{i-1} and V_iV_{i+1} contains only points from the interior of the polygon. It is no hard exercise to convince oneself that the so defined convexity of vertices is the same as the one we considered above.

Another useful result that follows from the Jordan theorem is that, given a point P, we can determine whether it belongs to the interior or exterior of a polygon using the following simple procedure: construct a ray with P its starting point and find out whether it intersects the line $V_1V_2\ldots V_nV_1$ at an odd or even number of points – these correspond to interior and exterior, respectively. Here we use the fact that the ray can only intersect the polygon at a finitely many points. In order to avoid indeterminacy or difficulty in determining intersections, the ray must be given such a direction that it does not pass through any of the vertices of the polygon. Again due to having finitely many of them, such a choice is always possible.

Triangulation

To triangulate a polygon means to decompose it into triangular pieces where the vertices of each triangle are those of the polygon. It is implied that the interior of a triangle is a subset of that of the polygon, the whole polygon is covered by triangles, and no two triangles overlap.

It follows that a triangulation, if possible, amounts to selecting a proper set of diagonals of the polygon. And it is clear that some polygons can be triangulated in more than one way – to observe this it suffices to consider a convex quadrilateral. However, it is not immediately obvious whether a polygon can be triangulated at all.

Possibility of triangulation, and in fact an algorithm for triangulating a polygon, follows from repeated construction of marked segments as we did for proving the polygonal variant of the Jordan curve theorem. After realizing that each such segment, being related in one of two ways to a convex vertex, lies entirely in the interior of the polygon and is therefore a diagonal, it remains to repeat constructing diagonals until all resulting sub-polygons are triangles. The discussion in the proof of the theorem guarantees that every polygon with more than three vertices has a diagonal.

If a certain triangulation of an n-gon produces t triangles, then this must have been achieved with constructing $t-1$ diagonals because each added diagonal increases by 1 the number of areal figures into which the polygon is decomposed, and before any diagonal is drawn there is only one figure – the polygon itself. On the other hand, if there are t triangles, then there must be $(3t-n)/2$ diagonals: each triangle has 3 sides, $3t$ in total, including the n sides of the n-gon and the diagonals, each counted twice because it is a side of two triangles. Equating $t-1$ to $(3t-n)/2$ yields $t = n-2$.

Thus no matter how a triangulation of an n-gon is done, there are exactly $n-2$ triangles obtained by drawing $n-3$ diagonals.

If a triangulation is considered a graph, in the graph-theoretic sense, with nodes at the vertices and an edge for each side and each diagonal of the polygon, then the dual graph has a node for each triangle, linking two nodes wherever the respective triangles share a side.

The dual graph is connected because the polygon itself is, and, as just pointed out, its edges are one less in number than the nodes, it is actually a tree – a connected graph with no loops. This relation between the numbers of nodes and edges is characteristic of trees and is easily established by induction.

Conclusion

We have presented a number of facts concerning general simple polygons. The presentation is sufficiently simple to be accessible to students in secondary school but requires commitment, diligence, and thoroughness from the audience. It combines geometry, combinatorial counting, a bit of graph theory, and offers a glimpse at topology.

Establishing the validity of the Jordan curve theorem for polygons, on the one hand, exemplifies the necessity for rigorous consideration of even most fundamental, albeit seemingly trivial matter (or precisely of such matter). On the other hand, it is indispensable for dealing with the rest of the topics, and thorough understanding of this dependence is another expected benefit of discussing such a subject at school.

We end by noting that there are other no less important and interesting topics related to polygons that are simple enough to be discussed in school, e.g. area and areal centre calculation. They deserve individual attention and hopefully will be the subject of at least one paper to follow this one.