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CERTAIN CLASSES OF FUNCTIONS WITH MISSING COEFFICIENTS

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Abstract. The aim of present paper is to derive coefficient bounds, distortion theorem and radius of convexity for a class of meromorphic convex functions.

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1. Introduction

Let Σ be the class of functions of the form:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in $U^* = \{z : 0 < |z| < 1\}$ having simple at $z = 0$ and residue one there.

We denote by $\Sigma(p)$ a class consisting of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n}, \quad p \in \mathbb{N}. \tag{1.2}$$

Let $\Sigma^*(p)$ be a class of the functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n}, \quad a_{p+n} \geq 0,\ p \in \mathbb{N}. \tag{1.3}$$

Clearly we have a relationship

$$\Sigma^*(p) \subseteq \Sigma(p) \subseteq \Sigma.$$

Given α $(0 \le \alpha < 1)$, a function $f(z) \in \Sigma$ is said to be in the class of meromorphic starlike functions of order α, denoted by $\Sigma^*(\alpha)$ if

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad |z| < 1. \tag{1.4}$$

Similarly for α $(0 \le \alpha < 1)$, a function $f(z) \in \Sigma$ is in the class of meromorphic convex functions of order α, denoted by $\Sigma_k^*(\alpha)$ if

$$-\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad |z| < 1. \tag{1.5}$$

Definition. A function $f \in \Sigma(p)$ is said to be in the class $\Sigma(\alpha, \beta, A, B)$ if it satisfies the condition:

$$\left|\frac{\dfrac{zf''(z)}{f'(z)} + 2}{B\left(1 + \dfrac{zf''(z)}{f'(z)}\right) + [B + (A-B)(1-\alpha)]}\right| < \beta, \tag{1.6}$$

where

$$0 \le \alpha < 1, \quad 0 < \beta \le 1, \quad -1 \le A < B \le 1, \quad 0 < B \le 1. \tag{1.7}$$

Let us write

$$\Sigma^*(\alpha, \beta, A, B) = \Sigma^*(p) \cap \Sigma(\alpha, \beta, A, B) \tag{1.8}$$

We note that similar types of classes were studied rather extensively by Bajpai [1], Goel and Sohi [2] and Srivastava et al. [3].

In the present paper we obtain coefficient inequalities and a distortion theorem for the class $\Sigma^*(\alpha, \beta, A, B)$. Also we obtain the radius of convexity.

2. Coefficient Inequalities

Theorem 2.1. *Let the function $f(z)$ defined by* (1.2) *be analytic in U^*. If*

$$\sum_{n=0}^{\infty}\{(p+n+1)+\beta[B(p+n)+(B-A)\alpha+A]\}(p+n)|a_{p+n}| \le \beta(B-A)(1-\alpha) \tag{2.1}$$

then $f(z)\in\Sigma(\alpha,\beta,A,B)$.

Theorem 2.2. *Let the function $f(z)$ defined by* (1.3) *br analytic in U^*, then $f(z)\in\Sigma^*(\alpha,\beta,A,B)$ if and only if* (2.1) *is satisfied.*

Theorem 2.1 and Theorem 2.2 are proven by the application of a technique similar to the one, used by Uraleggadi and Ganigi [4].

Corollary. Let function $f(z)$ defined by (1.3) be in the class $\Sigma^*(\alpha,\beta,A,B)$. Then

$$|a_{n+p}| \le \frac{(B-A)(1-\alpha)}{(n+p)\{(n+p+1)+\beta[B(n+p)+(B-A)\alpha+A]\}}z^{p+n}$$

where equality holds for the function

$$f_{p+n}(z) = \frac{1}{z}+\frac{(B-A)\beta(1-\alpha)}{(n+p)\{(n+p+1)+\beta[B(n+p)+(B-A)\alpha+A]\}}z^{p+n}. \tag{2.2}$$

3. A Distortion Theorem

Theorem 3.1. *Let the function $f(z)$ defined by* (1.3) *be in the class $\Sigma^*(\alpha,\beta,A,B)$. Then for $0<|z|=r<1$*

$$\frac{1}{r}-\frac{(B-A)\beta(1-\alpha)r^p}{p\{(p+1)+\beta[Bp+A+(B-A)\alpha]\}} \le |f(z)| \le \frac{1}{r}+\frac{(B-A)\beta(1-\alpha)r^p}{p\{(p+1)+\beta[Bp+A+(B-A)\alpha]\}} \tag{3.1}$$

where equality holds for the function

$$f_p(z) = \frac{1}{z} + \frac{(B-A)\beta(1-\alpha)}{p\{(p+1)+\beta[Bp+(B-A)\alpha+A]\}} z^p \tag{3.2}$$

and

$$\frac{1}{r^2} - \frac{\beta(B-A)(1-\alpha)}{\{(p+1)+\beta[Bp+A+(B-A)\alpha+A]\}} r^{p-1} \le |f'(z)| \tag{3.3}$$

$$\le \frac{1}{r^2} + \frac{\beta(B-A)(1-\alpha)}{\{(p+1)+\beta[Bp+(B-A)\alpha+A]\}} r^{p-1}$$

where equality holds for the function $f_p(z)$ *given by* (3.2) *at* $z = \pm r$.

Proof. In view of Theorem 2.2 we have

$$\sum_{n=0}^{\infty} |a_{p+n}| \le \frac{(B-A)\beta(1-\alpha)}{p\{(p+1)+\beta[Bp+(B-A)\alpha+A]\}}, \tag{3.4}$$

Thus for $0 < |z| = r < 1$

$$|f(z)| \le \frac{1}{r} + \sum_{n=0}^{\infty} |a_{p+n}| r^{p+n} \le \frac{1}{r} + r^p \sum_{n=0}^{\infty} |a_{p+n}|$$

$$\le \frac{1}{r} + r^p \frac{(B-A)\beta(1-\alpha)}{p\{(p+1)+\beta[Bp+A+(B-A)\alpha]\}}$$

$$|f(z)| \ge \frac{1}{r} - \sum_{n=0}^{\infty} |a_{p+n}| r^{p+n}$$

$$|f(z)| \ge \frac{1}{r} - r^p \sum_{n=0}^{\infty} |a_{p+n}| \ge \frac{1}{r} - \frac{r^p(B-A)\beta(1-\alpha)}{p\{(p+1)+\beta[Bp+A+(B-A)\alpha]\}}$$

which together yield (3.1).

It follows from Theorem 2.2 that

$$\sum_{n=0}^{\infty} (n+p)|a_{n+p}| \le \frac{\beta(B-A)(1-\alpha)}{\{(p+1)+\beta[Bp+(B-A)\alpha+A]\}}.$$

Hence

$$|f'(z)| \le \frac{1}{r^2} + \sum_{n=0}^{\infty}(p+n)|a_{p+n}|r^{p+n-1} \le \frac{1}{r^2} + r^{p-1}\sum_{n=0}^{\infty}(p+n)|a_{p+n}|$$
$$\le \frac{1}{r^2} + \frac{\beta(B-A)(1-\alpha)}{\{(p+1)+\beta[Bp+(B-A)\alpha+A]\}}$$

$$|f'(z)| \ge \frac{1}{r^2} - \sum_{n=0}^{\infty}(p+n)|a_{p+n}|r^{p+n-1} \ge \frac{1}{r^2} - r^{p-1}\sum_{n=0}^{\infty}(p+n)|a_{p+n}|$$
$$\ge \frac{1}{r^2} - \frac{\beta(B-A)(1-\alpha)}{\{(p+1)+\beta[Bp+(B-A)\alpha+A]\}}$$

which together yield (3.3).

It can be easily seen that the function $f_p(z)$ defined by (3.2) is extremal for Theorem 3.1.

Theorem 3.2. *If $f(z)$ given by* (1.3) *is in the class $\Sigma^*(\alpha,\beta,A,B)$, then $f(z)$ is convex in the disk*

$$0 < |z| = r = r(\alpha,\beta,A,B)$$
$$= \inf_n \left(\frac{n+p+1+\beta[B(n+p)+(B-A)\alpha+A]}{(B-A)\beta(1-\alpha)(p+n+2)}\right)^{\frac{1}{p+n+1}}$$

The result is sharp for the function given by (2.2).

The proof of Theorem 3.2 is standard.

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НЯКОИ КЛАСОВЕ ОТ ФУНКЦИИ С ЛИПСВАЩИ КОЕФИЦИЕНТИ

Донка Пашкулева

Резюме. Целта на представената статия е да даде коефициентни оценки, теореми за ръста и радиус на изпъкналост за за един клас от мероморфни изпъкнали функции.