EXPERIMENTING WITH DIFFERENT APPROACHES TO TEST COLINEARITY OF POINTS

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Abstract: *The subject of collinearity of 3 points on a line is challenging for 7^{th}- grade students. This article discusses the possibilities to demonstrate different approaches for solving collinearity of three points on a line: straight angle, axiom on the uniform mapping of an angle on a half-plane, parallel axiom, vector method, and homothetic features. The experiment was held using GEONExT to demonstrate by visual methods collinearity of three points on a line a district type of problems.*

Keywords: collinear points, experimental problems
2010 Mathematics Subject Classification: 97D40

By the seventh grade, a straight angle and the degree measure of an angle have already been defined [3]. From this property follows:

(1) *Points A, O and B lie on a line in that order if* $\sphericalangle AOB = 180^\circ$

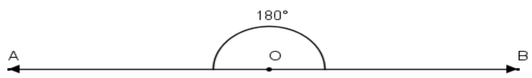


fig.1

The following problems are dedicated to the application of property (1) in establishing that three points lie on a straight line.

Problem 1 An acute-angled isosceles $\triangle ABC$ ($AC = BC$). Line ℓ halves exterior angle of $\triangle ABC$ at the apex C. The sequel to the height AH ($H \in BC$) of $\triangle ABC$ intersects the line ℓ in a point P. If $CM\left(M \in AB\right)$ is the bisector of $\sphericalangle ACB$ and point Q is the middle of CP, prove that the points M, H and Q lie on a straight.

Solution: The condition $\sphericalangle CAB = \sphericalangle CBA = \alpha$ and $AM = MB$ (fig. 2). Therefore, HM and HQ are middles to the hypotenuse of the rectangular triangles $\triangle AHB$ and $\triangle CHP$ where $\sphericalangle MHB = \sphericalangle MBH = \alpha$ and $\sphericalangle QCH = \sphericalangle QHC = \alpha$. Then $\sphericalangle MHA = 90^\circ - \alpha$ and $\sphericalangle MHQ = \sphericalangle MHA + \sphericalangle AHC + \sphericalangle CHQ = 90^\circ - \alpha + 90^\circ + \alpha = 180^\circ$. From the collinear property (1) follows that the points M, H and Q lie on the same line.

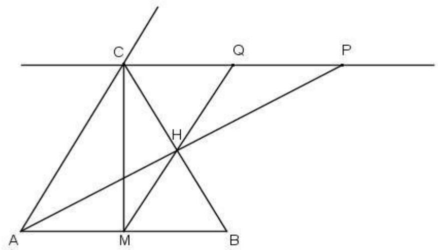


fig. 2

Problem 2 For rectangular $\triangle ABC$ ($\sphericalangle C = 90^\circ$) point C_1 is the middle of the hypotenuse AB, a point P is the medium segment of line CC_1. Lines CA and CB are symmetries to segments PN and PM. Prove that the points N, C and M lie on a straight line.

Proof: If CA is symmetry to the PN, it follows that $\sphericalangle NCA = \sphericalangle ACP = \alpha$. Analogically, $\sphericalangle PCB = \sphericalangle BCM = \beta$. As provided $\sphericalangle ACB = \alpha + \beta = 90^\circ$ where $\sphericalangle NCM = 2.\alpha + 2.\beta = 2.(\alpha + \beta) = 2.90^\circ = 180^\circ$. From property (1) it follows that the points N, C and M are on the same line.

Problem 3 A is a parallelogram *ABCD* ($\sphericalangle BAD < 90^\circ$) and *M* is the middle of *AD.* The line through apex *D*, is perpendicular to *DC*, crosses *AB* and *BC* in points *P* and *Q*. If *N* is the middle of the *BQ*, prove that the points *M, P* and *N* lie on the same line.

Proof: We mark $\sphericalangle DAB = \sphericalangle ABQ = \alpha$. Since *DP* is perpendicular to *DC, it* follows that *DP* is perpendicular to *AB,* i.e. triangles $\triangle APD$ and $\triangle BPQ$ are rectangular, and *PM* and *PN* are respective medians to their hypotenuses. Then $\sphericalangle MAP = \sphericalangle APM = \alpha$, $\sphericalangle QPN = \sphericalangle NQP = 90^\circ - \alpha$. Therefore $\sphericalangle MPN = \sphericalangle MPA + \sphericalangle APQ + \sphericalangle QPN = \alpha + 90^\circ + 90^\circ - \alpha = 180^\circ$. From property (1) we obtain that the points *M, P* and *N* lie on a straight line.

The following axiom is well known: *If two congruent angles with a common side are mapped on a half-plane whose contour is determined by the common side, the other sides of the angles coincide* [3].

So we get the following property:

(2) *For angles $\sphericalangle AOB = \sphericalangle AOC$, if shoulders OB and OC coincide, or points B, O and C lie on a straight line.*

Problem 4 For convex quadrilateral *ABCD* it is known that *AC* = *BD*, $\sphericalangle ADB = \sphericalangle CAB$ and $\sphericalangle ABD = \sphericalangle CAD + \sphericalangle ADC$. Find the size of $\sphericalangle BAD$.

Solution: From the fact that $\sphericalangle ABD$ is the sum of two angles in $\triangle ACD$, follows that it is equal to the external angle at the apex *C* (fig. 4). On *DC,* the point *F* is such that *CF* = *AB (C* is between *D* and *F).* Then, $\triangle ABD \cong \triangle FCA$ because *BD* = *AC* (by condition), *AB* = *CF* (by construction) and $\sphericalangle ABD = \sphericalangle CAD + \sphericalangle ADC = \sphericalangle ACF$ so we get $\sphericalangle CAF = \sphericalangle ADB$ (corresponding angles in congruent triangles). But $\sphericalangle ADB = \sphericalangle CAB$ (by condition), where $\sphericalangle CAB = \sphericalangle CAF$. If we apply property (2), we find that the points *A*, B and *F* lie on the same line. Since identity is proven, it follows that $\sphericalangle BAD = \sphericalangle AFC$ and *AD* = *AF*. From the equality of angles it follows that $\triangle AFD$ is isosceles with congruent sides *AD* = *DF*. Then *AD* = *DF* = *AF*, i.e. $\triangle AFD$ is equilateral and $\sphericalangle BAD = \sphericalangle FAD = 60^{\circ}$. (Peterburg's olimpiad, 1998)

Known is the following theorem: *Through a point not on a given line, exactly one line can be drawn on the plane parallel to the given line* [3].

Hence we get the property:

(3) *If AO and BO are passing through a point O, and which are parallel to line a, which is not passing through point O, then the points A, O and C lie on a straight.*

Problem 5 Let AA_1 $(A_1 \in BC)$ and CC_1 $(C_1 \in AB)$ are medians in $\triangle ABC$. On AA_1 and CC_1 are selected the points *P* and *Q* such that A_1 is the middle of *AP* and C_1 is the *mid-CQ*. Prove that the points *Q, B* and *P* lie on the same line.

Proof: The diagonals in quadrangles *AQBC* and *ABPC* bisect each other, hence they are parallelograms. Then $QB \parallel AC$, $BP \parallel AC$ and from property (3) follows that the points *Q*, *B* and *P* are on the same line.

Problem 6 The points *M* and *N* are on the respective sides of *BC* and *CD* of parallelogram *ABCD.* If $AB \cap DM = E$ and $AD \cap BN = F$ prove that the points *E, C* and *F* lie on the same line.

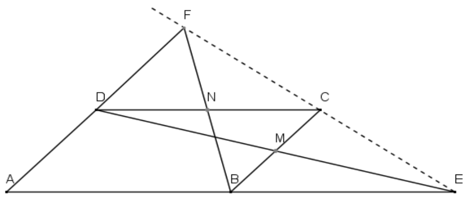


fig.3.

Proof: Consider $\triangle DNF$ and $\triangle CNB$ (see fig. 3). It is known that *DN* = *NC,* $\sphericalangle DNF = \sphericalangle CNB$ (top angles) and $\sphericalangle FDN = \sphericalangle NCB$ (bottom angles for $AF \parallel CB$, crisscrossed with *DC).* Therefore, $\triangle DNF \cong \triangle CNB$, consequently *FN* = *NB.* Since *DN* = *NC,* then the quadrilateral *DBCF* is a parallelogram. Similarly, we show that the quadrilateral *DBEC* is a parallelogram. So we get that $FC \parallel DB$ and

$CE \parallel DB$, which together with the property (3) shows that the points *C*, *F* and *E* are on the same line.

In considering the themes of congruent triangles, the midperpendicular properties are established [3] and is deduced that the three bisectors of interior angles of a triangle intersect at one point [3]. These properties will be used in the following three problems.

Problem 7 In rectangular $\triangle ABC$ $(\sphericalangle C = 90^\circ)$ it is known that $AC < BC$. On the hypotenuse *AB* is taken point *Q* so that *AC* = *AQ*. A perpendicular to *AB* is constructed from the point *Q*, intersecting the side *BC* at point *M*, and *O* is the intersection of the bisectors of $\sphericalangle ABC$ and $\sphericalangle ACB$. Prove that points *A*, *O* and *M* lie on the same line.

Proof: From $\sphericalangle ACM = \sphericalangle AQM = 90^\circ$. *AM* - common for both angles and *AC* = *AQ* (from the instructions) consequently $\triangle ACM \cong \triangle AQM$ (fig.4). Then $\sphericalangle CAM = \sphericalangle QAM$, i.e. *AM* is bisector of $\sphericalangle BAC$. Therefore, *AM* will pass through point *O*, i.e. points *A*, *O* and *M* lie on the same line.

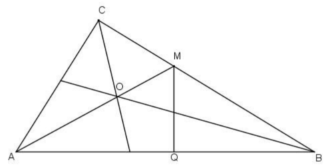


fig.4.

Problem 8 For isosceles $\triangle ABC$ $(AC = BC)$, upon the rays $CA^{\rightarrow}$ and $CB^{\rightarrow}$ are taken points *M* and *N* so that in *AB* = *AM* = *BN* and point *A* is between *M* and *C* and point *B* is between *N* and *C*. If *O* is the intersection of the *MB* and *AN*, point *P* is the intersection of the midperpendiculars *MB* and *AN*, prove that the points *C*, *O* and *P* lie on the line.

Proof: We mark $\sphericalangle BAC = \sphericalangle ABC = \alpha$ (fig.5). From isosceles triangles $\triangle MBA$ and $\triangle NAB$ we receive $\sphericalangle MBA = \sphericalangle NAB = \frac{\alpha}{2}$, i.e. $\sphericalangle OBA = \sphericalangle OAB = \frac{\alpha}{2}$ or *OA* = *OB*. Because *AP* and *BP* are midperpendiculars of the segments respectively *BM* and *AN*, $\sphericalangle PBA = \sphericalangle PAB = \frac{1}{2}\left(180^\circ - \alpha\right) = 90^\circ - \frac{\alpha}{2}$, i.e. $\triangle APB$ is an isosceles with sides *AP* and *BP*. Therefore, points *O*, *P* and *C* are equally spaced from the ends of the segment *AB* and each of them is lying on the line of symmetry.

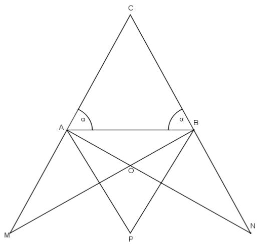


fig.5.

The next two problems will be based upon the fact that the diagonals of a parallelogram bisect each other [3].

Problem 9 In parallelogram *ABCD*, the diagonals intersect at *O*, points *P* and *Q* are chosen on sides *AD* and *BC*, respectively, so that *AP* = *CQ*. Prove that the points *P*, *O* and *Q* lie on the same line.

Proof: By condition, *AP* = *CQ* and $AP \parallel CQ$, from which it follows that the quadrilateral is a parallelogram *AQCP*. Since point *O* is the midpoint of the diagonal *AC*, then *O* is an environment and *PQ* diagonal, it follows that the points *P*, *O* and *Q* lie on the line.

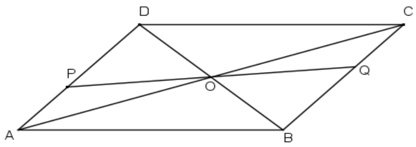


fig. 6.

Problem 10 In rectangle *ABCD* with intersection of the diagonals *O*, we build bisector BM $(M \in DC)$ of $\sphericalangle DBC$ and DL $(L \in AB)$ of $\sphericalangle ADB$ **.** Prove that the points *L*, *M* and *O* lie on the same line.

Clue: Establish first that the quadrilateral is a parallelogram *LBMD*. Then the middle of *BD*, which is *O*, is at the same time middle of *LM*, from where it follows that the points *L*, *M* and *O* lie on the line.

8th class syllabuses cover the definition of sum of vectors and difference of vectors, multiplication of a vector by a number and the following properties:

- **(4)** *If the points A, B and C satisfy the equality* $\overrightarrow{AB} = \lambda \overrightarrow{AC}$ *where* λ *is a number, then points A, B and C lie on the line.*
- **(5)** *If M is the middle of segment AB and O is a random point, then the equality* $\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$ *holds.* [5]

These two properties are valid:

(6) *In $\triangle ABC$ with centric G and an arbitrary point O is true equality* $\overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$.

(7) *In $\triangle ABC$ with orthocentre H and the centre of a circle circumscribed around the triangle A is true the equality of Hamilton* $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$.

Problems to be considered are divided into two groups:

- In the first group, some of the equations (5) (6) and (7) are valid and the equality (1) holds for the points in question;
- In the second group, vector equalities are sought for the points in question by introducing basic vectors.

Problem 11 Prove that for $\triangle ABC$ centroid *G*, orthocentre *H* and centre of the circle circumscribed around the triangle lie on the same line (Oiler's line).

Hint: From equations (6) and (7) it is found that $\overrightarrow{OH} = 3.\overrightarrow{OG}$.

Problem 12 In quadrilateral *ABCD* points *M* and *N* are middles of *AB* and *DC.* Point *P* is the midpoint of the segment *MN*, and *G* – centroid of $\triangle BCD$. Prove that points *A*, *P* and *G* lie on the same line.

Hint: From (5) and (6) follows that $\overrightarrow{AP} = \frac{1}{2}(\overrightarrow{AM} + \overrightarrow{AN}) = \frac{1}{4}(\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{AC}) = \frac{3}{4}\overrightarrow{AG}$.

Problem 13 Let *O* be any point from the plane of $\triangle ABC$ not belonging to the lines or their extensions, and the points *P*, *Q* and *R* are centroids of triangles $\triangle AOB$, $\triangle BOC$ and $\triangle COA$. Prove that the point *O* and centroids of $\triangle ABC$ and $\triangle PQR$ lie on the same line.

Hint: If *G* and G_1 are centroids for $\triangle ABC$ and $\triangle PQR$, respectively, use (6) and prove $\overrightarrow{OG_1} = \frac{1}{3}(\overrightarrow{OR} + \overrightarrow{OP} + \overrightarrow{OQ}) = \frac{2}{9}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) = \frac{2}{3}\overrightarrow{OG}$.

Let us consider the problems from the second group.

Problem 14 *ABCD* is a parallelogram. Points *M* and *N* are on sides *AB* and on the diagonal *AC* and such that $AM : MB = m : n$ and $AN : NC = m : (m+n)$. Prove that the points *D*, *M* and *N* lie on the same line.

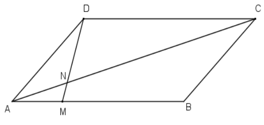


fig.7a

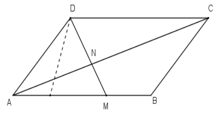


fig.7b

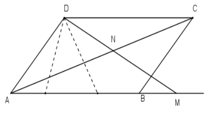


fig.7c

This type of problems are suitable in visualizing the solutions by use of information technology. We made an experiment with 9. grade students from OMG "Akad.K.Popov" - Plovdiv. The first lesson they had to solve the problem with specific values of *m* and *n*, using the vector method. The next lesson they got acquainted with *GEONExT* system and their new assignment was to see what were the changes in defining various values for the parameters. The teacher guided the students to establish relationships for colinearity of three points. The lesson was extremely successful and showed another approach to solving this type of tasks. Several cases were tested and analysis was made based on a relation scheme from a particular case to the general one[1]. Visibility and the power of *GEONExT* impressed many of the students.

The image of a plane onto itself is such that point X corresponds to a point X^I *such as* $\overrightarrow{OX^I} = k.\overrightarrow{OX}$ *where k ≠ 0 and is freely chosen, and O is a randomly chosen point, is called homothetic with centre O and coefficient k* [4].

Problem 15 For trapezoid $ABCD$ $(AB \parallel CD)$ diagonals intersect at point *O*, and extensions of its legs concur at point F. If the *M* and *N* are midpoints of the bases *AB* and *CD*, demonstrate that the points *M*, *N*, *O* and *E* lie on the line. (Steiner's problem)

Hint: Explore homotheties $h_1(O; A \to C)$ and $h_2(E; A \to D)$, then use the above property.

Problem 16 Point *O* inside $\triangle ABC$ is the intersection of three congruent circles, each of which lies inside this triangle and touches two of its sides. Prove that the point *O* and the centre of the inscribed and circumscribed circles with regard to $\triangle ABC$ lie on a line (from materials of XXII MOM).

Hint: If O_1, O_2 and O_3 are the centres of three circles, prove that $O_1O_2 \parallel AB$, $O_2O_3 \parallel BC$, $O_3O_1 \parallel CA$ and $O_1O = O_2O = O_3O$, i.e. *O* is the centre of the circle circumscribed around $\triangle O_1O_2O_3$. Moreover, the points O_1, O_2 and O_3 lie on the bisector of the interior angles of $\triangle ABC$ (why?). If $AO_1 \cap BO_2 = L$, then *L* is the centre of the circle inscribed in $\triangle ABC$. From homothety $h(L; O_1 \to A)$ it follows that $\triangle O_1O_2O_3$ is projected in $\triangle ABC$ and the centres of the circles circumscribed around them and the centre of homothety *L* lie on the same line.

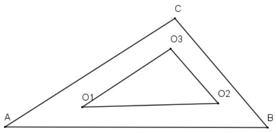


fig.8

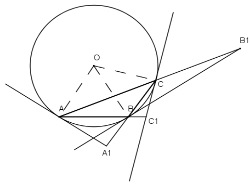
9th classes study Menelai's theorem [2]:

(8) On the sides BC, AC and AB in triangle $\triangle ABC$ are taken points A_1, B_1 and C_1, exactly one or three of them are external to the respective sides of the triangle. The points A_1, B_1 and C_1 lie on a line exactly when the equality $\frac{AC_1}{C_1B}.\frac{BA_1}{A_1C}.\frac{CB_1}{B_1A}=1$ is fulfilled

Problem 17 Prove that the bisectors of the exterior angles of an scalene triangle intersect the extensions of opposite sides of the triangle at points located on the same line.

Problem 18 Tangents to the circle circumscribed around a scalene, cross the extensions of the opposite sides in points lying on the same line.

Hint: Establish that $\triangle A_1AC \sim \triangle A_1BA$, fig.9 where $\frac{CA_1}{AA_1}=\frac{AA_1}{BA_1}=\frac{AC}{AB}\Rightarrow BA_1=\frac{AB.AA_1}{AC}$ and $CA_1=\frac{AC.AA_1}{AB}$. Then $\frac{BA_1}{CA_1}=\frac{AB^2}{AC^2}$, $\frac{CB_1}{B_1A}$



and $\frac{AC_1}{C_1B}$ apply Menelai's theorem.

fig. 9

Problem 19 Adjacent lines intersect BC, CA and AB of $\triangle ABC$ or their extensions in points A_o, B_o and C_o. Prove that A_1, B_1 and C_1 on segments AA_o, BB_o and CC_o, respectively, lie on the same line.

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