MATHEMATICAL MODELING WITH REAL DATA FOR UNDERGRADUATE STUDENTS

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ABSTRACT

For several years I have been introducing some basic concepts of mathematical modeling to undergraduate students from a Bulgarian university. The examples on the elections' theory and practice are accessible and interesting to the entire audience. Driven by the idea to obtain the results of the elections for the European Parliament, the students analyze the documents describing the Hare-Niemeyer method. The use of real or close to real data gives authenticity to the problems. The inquiry-based learning helps the students develop their reasoning, problem solving skills, and creativity. Problem situations have a strong impact on students' mathematical literacy and encourage them to study mathematics.

Key Words: Mathematics education of undergraduate students, Mathematical modeling of elections, Method of Hare-Niemeyer, Rational numbers rounding, Largest remainder, Data processing, Data constructing.

INTRODUCTION

Reflections from undergraduate students I have taught over the years have proved that among the most interesting curricular topics is the one on distribution of seats in parliamentary elections[1]. Although solving word problems at school has already given students some experience with mathematical models, modeling of elections seems quite different to them. Still, they can successfully model many situations in electoral practice using their basic knowledge of rounding rational numbers in a new algorithmic context.

THE CONCEPT OF TRANSITION FROM RATIONAL QUOTIENTS TO INTEGER NUMBERS OF THE PARLIAMENTARY SEATS

The method of *Hare-Niemeyer* for processing the election votes and distribution of seats (also known as the *Largest remainder method*) was adopted in the Republic of Bulgaria in 2007. A detailed description of its algorithm is promulgated in Bulgarian State Gazette[2]. According to it, first an important integer bound named a *national electoral quota* (NQ) is to be found. It determines the political parties which participate in the distribution of seats. Another important integer is called a *Hare quota* (HQ). It shows the number of votes which elect a single parliamentary seat. On the basis of HQ and the respective number of votes, the number of seats for each political party is to be calculated. A key moment is that the ratios of the votes in favor of each political party to the Hare quota are generally rational numbers, while the numbers of parliamentary seats the parties win in the elections are integer. Therefore, a grounded procedure for transition from the rational quotients to the integer numbers of seats is to be applied, which itself is the essence of the Hare-Niemeyer method.

When the undergraduate students in my classes use the method of Hare-Niemeyer for the first time, they feel perplexed: *rounding* of rational numbers to integers they are familiar with *is not* what the method requires. To emphasize the *remainders*' *approach* of the Hare-Niemeyer method, I have designed the following example:

Problem situation 1. Apply the method of Hare-Niemeyer to allocate 17 seats among the political parties **A**, **B**, **C**, **D**, and **E**, if the number of the votes they have received is:

Party **A**	Party **B**	Party **C**	Party **D**	Party **E**	Total
175,000	520,000	640,000	430,000	235,000	2,000,000

The numerical data above is not random, but close to the real one. The total number of 2,000,000 votes is approximately equal to the number of Bulgarian citizens who voted in the European Parliament elections: in 2007, they were 1,937,696 (Central Election Committee, 2007) and in 2009 – 2, 576,434 (Central Election Committee, 2009). The number of seats allocated for Bulgaria in the 2009 European Parliament is 17, won by 6 political parties. To avoid political predilections or unintended analogies with the real participants in Bulgarian elections, the examples herein discussed have been designed for 5 political parties.

A surprise for the students working on *Problem situation 1* is the special rounding of the ratio between the total number of votes and the number of seats. According to Bulgarian legislation, the nearest integer *exceeding* that ratio is to be taken to get the Hare quota[2]. The ratio of 117,647.0588 which students regularly round to 117,647 is thus to be rounded to 117, 648. Therefore, HQ equals 117,648 votes.

The students' "solution", shown in *Table 1*, illustrates the effect of *rounding the ratios* between the numbers of votes to HQ *instead of* considering their decimal parts and successively *taking the largest* one (the *remainder*). Although algebraically correct, such a procedure is not in accordance with the Hare-

Niemeyer method. As a result, the total number of the allocated seats comes up to 16 and not to 17.

This error makes students be more precise when following the algorithm of the Hare-Niemeyer method. At the first step, each political party receives a number of seats equal to the *integer part* of the ratio between the votes and HQ. In general, this procedure may leave several unallocated seats which are distributed at the next step. The party, whose ratio has the largest decimal part (remainder), takes one additional seat. The procedure continues until all seats have been distributed. If at the final step two or more parties have equal remainders, according to the law, a lot is drawn [2]. Such a situation is highly attractive to the students and has been thoroughly examined in class (Gortcheva, 2010).

Table 1. Students' *misunderstanding* of rounding and choosing the largest remainder

	Party **A**	Party **B**	Party **C**	Party **D**	Party **E**	Total
Votes	175,000	520,000	640,000	430,000	235,000	2,000,000
Votes/HQ	1.4875	4.4200	5.4400	3.6550	1.9975	17.0000
"*Solution*"	1 seat	4 seats	5 seats	4 seats	2 seats	16 seats

The correct solution to *Problem Situation 1* is given in *Table 2*. It shows that a rational number like 1.4875, whose decimal part is less than 0.5 has been rounded to 2, but not to the nearest integer 1. The explanation students accept is that the arranged in a descending order sequence of remainders 0.9975; 0.6550; 0.4875; 0.4400; 0.4200 has to be cut after the third term because three additional seats are to be distributed. In another situation the same sequence of remainders might be cut after the first term. Then a number like 3.6550, whose decimal part is greater than 0.5, is to be rounded to 3, but not to the nearest integer 4. Therefore, the concept of rounding rational numbers to the nearest integers taught in the middle and high school is not equivalent to the method of Hare-Niemeyer.

Table 2. Distribution of 17 seats by the method of *Hare-Niemeyer*

	Party **A**	Party **B**	Party **C**	Party **D**	Party **E**	Total
Votes	175,000	520,000	640,000	430,000	235,000	2,000,000
Votes/HQ	1.4875	4.4200	5.4400	3.6550	1.9975	17.0000
Integer part	1	4	5	3	1	14
Remainder	**0.4875**	0.4200	0.4400	**0.6550**	**0.9975**	3.0000
More seats	1			1	1	3
Seats	2	4	5	4	2	17

INCREASING THE TOTAL NUMBER OF SEATS

Under the Treaty of Lisbon, signed on December 13, 2007, the number of seats at the European Parliament is to be increased from 736 to 751. This allows Bulgaria to have one more seat in the future – 18 in total. According to Bulgarian law[2], it goes to the party with the largest unused remainder. However, the students should be aware that re-distributing a greater number of seats may cause the undesired effect that some of the political parties receive fewer seats than under the previous distribution. Cases of the sort are rare, but not impossible:

Problem situation 2. Use the data from *Problem Situation 1* and the method of *Hare-Niemeyer* to allocate 18 seats among the parties **A**, **B**, **C**, **D**, and **E.**

In this case, the Hare quota is equal to 111,112 votes. The results of distributing the 18 seats applying the Hare-Niemeyer method are given in *Table 3*. Comparison to the results in *Table 2* shows that Party **A** has lost one of the seats it got under the previous distribution. Such an odd effect, known as the *Alabama Paradox*, was unexpected to my audience, but highly beneficial to its mathematical literacy.

Table 3. Distribution of 18 seats by the method of *Hare-Niemeyer*

	Party **A**	Party **B**	Party **C**	Party **D**	Party **E**	Total
Votes	175,000	520,000	640,000	430,000	235,000	2,000,000
Votes/HQ	1.5750	4.6800	5.7600	3.8700	2.1150	18.0000
Integer part	1	4	5	3	2	15
Remainder	0.5750	**0.6800**	**0.7600**	**0.8700**	0.1150	3.0000
More seats		1	1	1		3
Seats	1	5	6	4	2	18

The heuristic practice of solving more problems with varied numerical data has helped the students gain useful mathematical experience (Grozdev, 2010, pp. 185-203). It has allowed them to conclude that practically, the *Hare-Niemeyer* method is not fully proportional and none of the legislators, politicians, or voters can expect the distributing of seats to be flawless.

THE POWER OF ONE VOTE

To make students feel personally involved with mathematical modeling I have designed a special task which demonstrates them the importance of one single vote. They are to construct a special distribution of votes received by 5 political parties, such that the shortage of only one vote prevents a party from winning one more parliamentary seat. Here is how such a task can be formulated:

Problem situation 3. Statistics show that 2,000,000 people have voted for the five political parties **A**, **B**, **C**, **D**, and **E** in the European Parliament elections. The number of parliamentary seats is 17 and the method of *Hare-Niemeyer* is used to allocate them among the parties. Construct a table of the votes distributed among the five parties, whereby the shortage of one vote for Party **A** causes it to win one seat less than Party **B**. Check your results by the *Hare-Niemeyer* method.

This problem situation proves the students that during democratic elections, the vote of any of them can cause a decisive change in society and vice versa, "*the ignorance of one voter in a democracy impairs the security of all*" (Kennedy, 1963). Popat and Powell (2007) support another attitude: "in the political context, indulging irrational beliefs in the voting booth has an expected cost of nearly zero, since one vote is not going to change the outcome of the election" (p. 125). The opposite perspectives to the same issue challenge my students to find their own standpoint to it. The process of getting a solution has led them to a heuristic search, which itself is an opportunity for developing their mathematical skills and intuition. As Grozdev (2007) points out, to successfully manage with such a nonstandard task, the students "perform conscientious analysis of the problem situation, rationalize the initial data and task (what is sought), and purposefully vary visible and hidden properties of all elements of the problem condition. Then the chance of insight increases and finding a path to solution occurs" (p. 172). As a result of the students' inquiry, several solutions have been found and written on the board. Here are two of them (*Tables 4-5*):

Table 4. Students' distribution **No. 1** of 17 seats on "The power of one vote" task

	Party **A**	Party **B**	Party **C**	Party **D**	Party **E**	Total
Votes	647,064	647,065	352,927	235,296	117,648	2,000,000
Votes/HQ	5.4999	5.5000	2.9998	2.0000	1.0000	16.9997
Integer part	5	5	2	2	1	15
Remainder	0.4999	**0.5000**	**0.9998**	0.0000	0.0000	1.9997
More seats		1	1			2
Seats	5	6	3	2	1	17

Table 5. Students' distribution **No. 2** of 17 seats on "The power of one vote" task

	Party **A**	Party **B**	Party **C**	Party **D**	Party **E**	Total
Votes	282,355	282,356	494,105	823,534	117,650	2,000,000
Votes/HQ	2.3999	2.4000	4.1998	6.9999	1.0000	16.9996
Integer part	2	2	4	6	1	15
Remainder	0.3999	**0.4000**	0. 1998	**0.9999**	0.0000	1.9996
More seats		1		1		2
Seats	2	3	4	7	1	17

The construction of a set of data with a specific property helps students significantly raise their confidence regarding data analysis and interpretation. Starting with the trial-and-error method, they continue with the construction of proper remainders and work backward to obtain the number of votes. In contrast to *Problem situations 1* and *2,* which require straight processing of the tables of votes, *Problem situation 3* provides a different experience for the students, since no data is available for processing. Therefore, it creates an opportunity to introduce my students to the idea of an *inverse problem* as well as of *multiple solutions*.

Problem situation 3 presents the students with an interesting pattern on rational numbers' decimal representation. By comparing the number of votes which favor Party **E** in the solutions shown in *Tables 4* and *5*, they have noticed that the ratios of 117,648 to 117,648 and of 117,650 to 117,648 are not distinguishable, if rounded to the nearest ten thousandths.

Complexity of approaches this task has brought in class appeals to the students in both mathematics and non-mathematics majors. One of the students sincerely exclaimed out loud: "*So far I have constructed medians, bisectors, circles, triangles, Lego, but not data*". The strong impression *Problem situation 3* has left makes me believe that my audience is already knowledgeable about the social power of voting. Comparison between the *D'Ondt* and *Hare-Niemeyer* methods also raises undergraduate students' genuine interest to the theme (Gortcheva, 2010). I am inclined to think that mathematical modeling of elections can be also a subject to high school students' inquiry in the frame of mathematical or interdisciplinary projects. Problem situations like these satisfy the criteria developed by Marasheva-Delinova (2010) and are easy to approach by pupils more interested in the humanities.

STUDENTS' ACHIEVEMENTS

In my teaching practice I regularly use the methods of qualitative research to adjust the curriculum according to the students' needs. Triangulation of data sources (documents, observations, and interviews) is a tool to increase accuracy and credibility of results (Patton, 2005, p. 91). Holistic approach and analytical perspectives to the educational process help me design mathematical problems, which sound attractively to the audience and match their knowledge (Gortcheva, 2010). Activation of undergraduate students' personal reflexivity can be used to control the effectiveness of the class work (Dimova & Chehlarova, 2005). My analysis of students' midterms, grades, questionnaires, and reflections, as well as of their questions, answers, and reactions during our communications in class and after the classes has made modeling of elections with real or close to real data one of the students' favorite topics. The theme boosts their interest in mathematics and positively changes their attitude towards learning. Students' grades also change: from quite modest before working on the topic to the highest possible afterwards. This effect encourages me to introduce more approaches to mathematical modeling in my classes, as linear programming (Kelevedjiev & Gortcheva, 2010), dynamic optimization (Sendov, 2010), and game theory (Brams, 2004, pp.101-121).

CONCLUDING REMARKS

My aim in teaching mathematics has always been that mathematical concepts make sense to the undergraduate students. Quotations by world-known scholars or proponents of science break the ice and at the same time give unusual perspectives to mathematical ideas (Gortcheva, 2009). As a final note to the mathematical models of electoral situations presented in class I share with my students the thought of the influential British playwright Sir Tom Stoppard (1972): "*It's not the voting that's democracy*; *it's the counting*" later on clarifying that "*Mathematics is not simply the technique of counting*".

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Notes:

[1] The course is named *Classical and modern ideas in mathematics* and is taught in New Bulgarian University in Sofia;

[2] Method for determining the results of voting for the Bulgarian representatives in the European Parliament. (2009, April 10). *The State Gazette* No. 27, pp. 124-128 (in Bulgarian).

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